

**Pricing Barrier Options using Monte Carlo simulations**

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**Abstract:**

The pricing of path dependent options such as barrier options is not straight forward using analytical formulae. The objective of this project is to compare pricing of the barrier options using standard Black Scholes model and Monte Carlo simulations. This project compares the four variance reduction techniques such as Antithetic Variates, Control Variates, Conditional Monte Carlo, and Importance Sampling with crude Monte Carlo simulation in pricing the barrier options and the convergence with the analytical solution as we increase the number of simulations. Also, we look at the impact of barrier price on the different variance reduction methods.

*Keywords: barrier option, Black-Scholes model, Monte Carlo simulations, variance reduction*    
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**1. Introduction**

Options are one of the highly traded financial derivatives in the market. Exchange traded options are mostly plain vanilla options. The more complex path dependent options which are also commonly known as exotic options are traded over the counter (OTC). The exotic options have non- standard features and are designed to cater the needs of a particular investor or client. Barrier options are a type of exotic options which depends on the price of the underlying asset reaching a pre-specified barrier level. Barrier options behave like European options apart from the fact that the options vanish or comes into existence once the price of the underlying asset hits the barrier level. The path dependency and the barrier level features of a barrier option gives us more options than the standard options for hedging.

Plain vanilla options can be priced easily using the Black-Scholes (BS) model considering that the price of the underlying asset follows a Geometric Brownian Motion (GBM). Barrier options can also be priced using the BS model though it is not straight forward, but we can have a closed form solution. There are various models, both closed form and numerical methods to price barrier options. One such numerical method to price barrier option is Monte Carlo simulation. It is a mathematical technique which uses concepts of random variables and probability distributions to generate a distribution of required outcome values. The accuracy of the Monte Carlo simulation method can be increased by increasing the number of simulations, but we should have a cap on the number of simulations because of the computational time it takes to run the simulations. The other method to improve the accuracy is by using different variance reduction techniques. The variance reduction method reduced the standard error of the Monte Carlo simulation with a particular number of simulations. We discuss four variance reduction methods namely, antithetic variates, control variates, condition Monte Carlo and the conditional expectation and importance sampling method.

The aim of this paper is to price the up-and-out call option using standard BS model and Monte Carlo simulations using four different variance reduction methods. The remainder of this paper is organized as follows: Section 2 presents the Empirical Methodologies whereby the concepts of pricing using BS model, Monte Carlo simulations, and variance reduction methods are analyzed. While the results are discussed in Section 3 and Section 4 concludes and recommends further work.

**2. Empirical Methodology**

Options are categorized into mainly two types based on the exercise time or time to maturity: European and American. An European option can only be exercised at the time to maturity whereas an American option can be exercised at any point of time until maturity. The price of an European call option, denoted by c, at time of initiation of the contract is,

Where is the price of the underlying at maturity T, K is the exercise price and r is the rate of interest. Similarly, the price of a European put option, denoted by p, at time of initiation of the contract is,

**2.1 Black-Scholes Model**

The benchmark model to price the options where the underlying asset is a stock is developed by Black-Scholes (1973). The BS model is based on the assumption that underlying stock prices follows a Geometric Brownian Motion (GBM) diffusion process and under real-world probability measure P, it is defined as:

where is the constant drift and is the constant volatility of the diffusion process. The BS model assumes market is complete and there are no arbitrage opportunities. Since the market is complete and there are no arbitrage opportunities, there exists a risk neutral measure. The GBM diffusion process for stock price under risk neutral probability measure is defined as:

where r is the risk-free interest rate. The BS model price of a plain vanilla European call option at initiation of the contract is:

where and

N(.) is the cumulative normal distribution function, and is the price of the underlying stock at initiation of the contract.

The price of the barrier options depends on the barrier level. Since barrier options are a type of path dependent options, the price of the barrier options depends on the path of the underlying stock. Barrier options are categorized as knock-in and knock-out barrier options. The knock-in options come into existence if the underlying stock price hits the barrier level, denoted by B whereas the knock-out options vanish if the underlying stock price hits the barrier level. The barrier options are further classified into up-and-in, up-and-out, down-and-in, and down-and-out based on the value of B, with respect to underlying stock price at initiation . If is greater than B at the point of initiation, we deal with up type of options and if is lesser than B, we deal with down type of options.

The price of an up-and-out call option for the case where the underlying stock price follows a GBM diffusion process, derived by Shreve (2004) is:

where,

b =

The parity between barrier call options is defined as,

**2.2 Monte Carlo simulations**

Monte Carlo simulations is a commonly used numerical method to price options when there isn’t any closed form solution for a particular model. In order to estimate the price of any option using Monte Carlo simulations, we have to generate simulations of price of the underlying stock price at maturity using its probability density function. Then, the option payoff for each simulation is calculated. Finally, the price of the option is calculated by averaging the discounted payoffs of all simulations. Under the BS model assumption that stock prices follow GBM, the integral form of the price of the underlying stock price at time t is given by

As we know that is a standard Brownian Motion and it is normally distributed with mean of zero and variance of t. Using this we can say that is lognormally distributed with parameters,

The discretized version of the equation (3) over the time interval (0, T) with a time step of can be written as follows:

where , is a standard normal random variable. Since Monte Carlo simulations is a discrete time valuation, we use the equation (4) to simulate the price of the underlying stock. The price of the barrier option depends on the path of the underlying path, so we generate the simulations of the path of underlying stock price in order to capture the time at which the path of underlying stock price hits the barrier level. Then we can calculate the price of the barrier option by averaging the discounted payoffs from all simulations.

**2.3 Variance Reduction methods**

The accuracy of the pricing barrier option using Monte Carlo simulations can be increased by increasing the number of time steps over the time interval (0, T). Another way to improve the accuracy is by reducing error or variance of the Monte Carlo simulation using variance reduction methods.

**2.3.1 Antithetic Variates**

The antithetic variates variance reduction method uses two negatively correlated random variables to reduce the variance. Let X and Y be two random variables with the same distribution and X = - Y. The variance of summation of X and Y can be written as,

Since our assumption is X and Y are negatively correlated, the covariance will be negative, and the variance will be lower compared to the case if X and Y are positively correlated.

As we know that is normally distributed with mean zero and variance t. and are both normally distributed with same parameters*.* . A pair of standard normal random variables are used to generate the simulations for paths of price of underlying stock price and then the average of discounted payoffs from both the simulations is averaged to get the estimate of the price of the barrier option. Since the payoff function of a call option is monotonically increasing then the covariance of payoffs generated using negatively correlated BMs will be negative always. Hence, the variance of the Monte Carlo simulation will be reduced using this method.

**2.3.2 Control Variates**

The control variates variance reduction method makes use of a control variate. Let X be a random variable and Y be another random variable with known expected value E(Y), which is correlated with X. In this case, Y is the control variate. Let Z be another random variable defined such that the expected value of Z has to be equal to expected value of X,

where c is some arbitrary value. The variance of Z is,

We will use Z as our random variable to run Monte Carlo simulations. The value of c to minimize the variance of Z is,

The unknown Cov (X, Y) and Var(Y) can be estimated using Monte Carlo simulations.

In order to calculate the price of the barrier option, we need to calculate the Monte Carlo simulations of the payoff function and then average the discounted payoffs from all the simulations. In order to use control variates reduction method, we need to find the covariance between the plain vanilla call option and the up-and-out call option. This is done by running a set of pilot replications. Then we can calculate the optimal value of using the equation (6). By utilizing equation (5), we can get a reduced variance for our Monte Carlo simulation of price of the up-and-out option by incorporating .

**2.3.3 Conditional Monte Carlo**

The conditional Monte Carlo variance reduction method is based on the parity relation in equation (2). The Monte Carlo simulation is stopped as soon as the price of the underlying stock hits the barrier level. As we know that, once the barrier level is reached, the barrier option behaves as a plain vanilla option. We calculate the price of up and in barrier call option using this method in order to calculate the price of the up-and-out call option eventually.

Let time step to be and if the price of the underlying stock hits the barrier at time instant index of j, then the option is activated at time, j\*. The barrier option will behave as a plain vanilla call option from time, j\*. We can use the BS model to calculate the payoff of plain vanilla call option at that particular time stamp. If the barrier is not hit over the time interval (0, T), the payoff of the barrier option is considered to be zero for that particular path. The price of the up and in barrier call option will be calculated by averaging the discounted payoffs over all Monte Carlo simulated paths.

**2.3.4 Importance Sampling**

The importance sampling Monte Carlo variance reduction technique is based on the concept of changing the probability distribution of the underlying stock price in order to increase the probability of underlying stock price hitting the barrier level. As we know that, to generate random samples of a random variable X using Monte Carlo simulations, we need the density of the random variable X. Let f(x) be the density of X and h(X) be a function of X. The expected value of h(X) using the density f(x) can be written as,

In order to change the distribution of random variable X, we need a new density function, denoted by g(x) such that the density, f(x) = 0 whenever g(x) = 0. We can rewrite equation (7) as follows:

The equation (8) implies that expected value of h(X) under density function f(x) is equal to the expected value of under density function g(x) where is called as likelihood ratio. The term is defined as . Now we have two distributions with two different densities f(x) and g(x). The expected value of h(X) under density f(x) is equal to the expected value of under density g(x). The variance of h(X) is given by,

Whereas the variance of is given by,

The difference between the two variances if given by,

In order to reduce the variance, we need the equation (9) to be positive. The reduction of variance using this method largely depends on the choice of density g(x). It has to be chosen such that,

Importance sampling and conditional Monte Carlo methods can be combined to reduce the variance further. The importance sampling method can be applied to the underlying stock price until it hits the barrier level and from them conditional Monte Carlo can be applied.

**3. Results**

The price of the up-and-out call option is calculated using the BS model and Monte Carlo using simulations. All the simulations were done by varying barrier price, B and initial underlying stock value, S0 and the number of steps over the time interval (0, T). Other variables are fixed at constant values such as K = 45, sigma = 0.2, r=mu=0.01, number of replications (NRepl) = 100, number of pilot runs (NPilot) = 100, time to maturity, T = 1 year.

Table 1: Summary of Monte Carlo simulation and BS model prices for up-and-out call option

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Initial value (S0)** | **Barrier level (B)** | **# Steps (n)** | **BS model price** | **Monte Carlo price** | **Standard error** |
| 50 | 60 | 100 | 1.83156 | 2.07872 | 3.52759 |
| 1000 | 1.77330 | 3.43863 |
| 10000 | 1.83474 | 3.30000 |
| 70 | 100 | 4.97539 | 5.61968 | 6.29994 |
| 1000 | 4.72172 | 5.73328 |
| 10000 | 4.81459 | 5.55974 |
| 80 | 100 | 6.51588 | 7.16523 | 8.48294 |
| 1000 | 6.89455 | 7.70314 |
| 10000 | 6.65083 | 7.33896 |
| 60 | 70 | 100 | 5.09621 | 5.45292 | 7.06505 |
| 1000 | 4.79433 | 6.77757 |
| 10000 | 4.90988 | 6.52689 |
| 80 | 100 | 10.87490 | 11.62231 | 9.09860 |
| 1000 | 10.43838 | 8.76473 |
| 10000 | 10.69296 | 8.34151 |

The BS model prices for different initial values and barrier levels is calculated. It is evident that as the barrier level is increased, the price of the up-and-out call option is increased. Since the probability of reaching the barrier decreases when the barrier level is increased, so the increase in the price of the option. The Monte Carlo simulations price gradually converges to the BS model price as the number of simulations are increased. Moreover, the standard error decreases slightly with the increase in number of simulations. Using Monte Carlo simulations, we can get a price very close to the BS model price as the number of simulations are increased but the computational time is always a concern.

Table 2: Comparison of Monte Carlo simulation and Antithetic variates prices for up-and-out call option

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **Initial value (S0)** | **Barrier level (B)** | **BS model price** | **# Steps (n)** | **Monte Carlo price** | **Standard error** | **Antithetic variates price** | **Standard error** |
| 50 | 60 | 1.83156 | 100 | 2.07872 | 3.52759 | 1.84643 | 1.95786 |
| 1000 | 1.77330 | 3.43863 | 1.87403 | 2.14167 |
| 10000 | 1.83474 | 3.30000 | 1.91937 | 2.28082 |
| 70 | 4.97539 | 100 | 5.61968 | 6.29994 | 4.99697 | 3.89742 |
| 1000 | 4.72172 | 5.73328 | 4.99949 | 4.26926 |
| 10000 | 4.81459 | 5.55974 | 4.97682 | 4.43740 |
| 80 | 6.51588 | 100 | 7.16523 | 8.48294 | 6.57960 | 4.70555 |
| 1000 | 6.89455 | 7.70314 | 6.55190 | 5.41537 |
| 10000 | 6.65083 | 7.33896 | 6.67148 | 5.48153 |
| 60 | 70 | 5.09621 | 100 | 5.45292 | 7.06505 | 5.01110 | 4.60452 |
| 1000 | 4.79433 | 6.77757 | 5.06361 | 4.73026 |
| 10000 | 4.90988 | 6.52689 | 5.02809 | 4.22548 |
| 80 | 10.87490 | 100 | 11.62231 | 9.09860 | 10.95928 | 6.18646 |
| 1000 | 10.43838 | 8.76473 | 10.92537 | 6.08849 |
| 10000 | 10.69296 | 8.34151 | 10.85265 | 5.85071 |

Table 2 compares the Monte Carlo simulation prices with the prices calculated using the Antithetic variates variance reduction method. We know that the Monte Carlo simulation price converges to the BS model price as the number of simulations are increased. But in case of Antithetic variates, we get prices which are very close to the BS model at very few number of simulations. As the number of simulations are increased, there is not much change in the value of the price using Antithetic variates. Moreover, the variance is less that the variance obtained from simple Monte Carlo simulations. Monte Carlo simulations using Antithetic variates variance method works better than simple Monte Carlo and gives better results even at fewer number of simulations.

Table 3: Comparison of Monte Carlo simulation and Control variates prices for up-and-out call option

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **Initial value (S0)** | **Barrier level (B)** | **BS model price** | **# Steps (n)** | **Monte Carlo price** | **Standard error** | **Control variates price** | **Standard error** |
| 50 | 60 | 1.83156 | 100 | 2.07872 | 3.52759 | 1.88950 | 3.21330 |
| 1000 | 1.77330 | 3.43863 | 1.86098 | 3.28556 |
| 10000 | 1.83474 | 3.30000 | 1.86193 | 3.16772 |
| 70 | 4.97539 | 100 | 5.61968 | 6.29994 | 4.99388 | 6.01693 |
| 1000 | 4.72172 | 5.73328 | 4.95600 | 6.17459 |
| 10000 | 4.81459 | 5.55974 | 4.95703 | 6.53852 |
| 80 | 6.51588 | 100 | 7.16523 | 8.48294 | 6.74329 | 7.59463 |
| 1000 | 6.89455 | 7.70314 | 6.39683 | 7.30285 |
| 10000 | 6.65083 | 7.33896 | 6.57012 | 6.99896 |
| 60 | 70 | 5.09621 | 100 | 5.45292 | 7.06505 | 4.94954 | 6.36021 |
| 1000 | 4.79433 | 6.77757 | 4.96682 | 6.37692 |
| 10000 | 4.90988 | 6.52689 | 5.10082 | 6.39663 |
| 80 | 10.87490 | 100 | 11.62231 | 9.09860 | 10.82015 | 8.95571 |
| 1000 | 10.43838 | 8.76473 | 10.82685 | 8.56452 |
| 10000 | 10.69296 | 8.34151 | 10.94365 | 9.75136 |

Table 3 compares the Monte Carlo simulation prices with the prices calculated using the Control variates variance reduction method. The results are kind of mixed, but it performs better than simple Monte Carlo when compared at same number of simulations for both. Overall, the error of Monte Carlo simulations using control variates is reduced compared to simple Monte Carlo and the prices converges to BS prices as number of simulations are increased.

Table 4: Comparison of Monte Carlo simulation and Conditional Monte Carlo prices for up-and-out call option

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **Initial value (S0)** | **Barrier level (B)** | **BS model price** | **# Steps (n)** | **Monte Carlo price** | **Standard error** | **Conditional Monte Carlo price** | **Standard error** |
| 50 | 60 | 1.83156 | 100 | 2.07872 | 3.52759 | 1.86130 | 7.46835 |
| 1000 | 1.77330 | 3.43863 | 1.84693 | 7.31573 |
| 10000 | 1.83474 | 3.30000 | 1.75994 | 7.27374 |
| 70 | 4.97539 | 100 | 5.61968 | 6.29994 | 5.05308 | 6.93056 |
| 1000 | 4.72172 | 5.73328 | 5.07333 | 6.86152 |
| 10000 | 4.81459 | 5.55974 | 4.83647 | 7.18649 |
| 80 | 6.51588 | 100 | 7.16523 | 8.48294 | 6.39258 | 4.92717 |
| 1000 | 6.89455 | 7.70314 | 6.39736 | 4.89368 |
| 10000 | 6.65083 | 7.33896 | 6.39796 | 4.88952 |
| 60 | 70 | 5.09621 | 100 | 5.45292 | 7.06505 | 5.10500 | 12.78559 |
| 1000 | 4.79433 | 6.77757 | 5.10369 | 12.52194 |
| 10000 | 4.90988 | 6.52689 | 5.14688 | 12.46988 |
| 80 | 10.87490 | 100 | 11.62231 | 9.09860 | 10.74864 | 12.41829 |
| 1000 | 10.43838 | 8.76473 | 10.83331 | 12.20555 |
| 10000 | 10.69296 | 8.34151 | 10.85164 | 12.16003 |

Table 4 compares the Monte Carlo simulation prices with the prices calculated using the Conditional Monte Carlo variance reduction method. Conditional Monte Carlo variance reduction method works better compared to simple Monte Carlo when the barrier level is away from initial price of the underlying stock. However, it performs worse when barrier level is close to the initial price of the underlying stock.

Table 5: Comparison of Monte Carlo simulation and combination of Conditional MC & Importance Sampling prices for up-and-out call option

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| **Initial value (S0)** | **Barrier level (B)** | **BS model price** | **# Steps (n)** | **Monte Carlo price** | **Standard error** | **Importance Sampling price** | **Standard error** |
| 50 | 60 | 1.83156 | 100 | 2.07872 | 3.52759 | 1.66485 | 7.57765 |
| 1000 | 1.77330 | 3.43863 | 1.85121 | 7.30964 |
| 10000 | 1.83474 | 3.30000 | 1.84053 | 7.24624 |
| 70 | 4.97539 | 100 | 5.61968 | 6.29994 | 4.79318 | 7.32635 |
| 1000 | 4.72172 | 5.73328 | 5.10110 | 6.76667 |
| 10000 | 4.81459 | 5.55974 | 5.09032 | 6.80328 |
| 80 | 6.51588 | 100 | 7.16523 | 8.48294 | 6.38987 | 4.94612 |
| 1000 | 6.89455 | 7.70314 | 6.39355 | 4.92089 |
| 10000 | 6.65083 | 7.33896 | 6.40064 | 4.87075 |
| 60 | 70 | 5.09621 | 100 | 5.45292 | 7.06505 | 4.95128 | 12.70442 |
| 1000 | 4.79433 | 6.77757 | 5.11365 | 12.50971 |
| 10000 | 4.90988 | 6.52689 | 4.93812 | 12.45778 |
| 80 | 10.87490 | 100 | 11.62231 | 9.09860 | 10.75898 | 12.39259 |
| 1000 | 10.43838 | 8.76473 | 10.83330 | 12.20546 |
| 10000 | 10.69296 | 8.34151 | 10.86039 | 12.13840 |

Table 5 compares the Monte Carlo simulation prices with the prices calculated using the combination of Conditional Monte Carlo and Importance Sampling variance reduction method. This method works very similar to what we have observed for Conditional Monte Carlo variance reduction method. The addition of Importance Sampling to the Conditional Monte Carlo have not added much towards the variance reduction.

**4. Conclusions/Recommendations**

This paper discusses the pricing of barrier options mainly up-and-out call option using the standard Black-Scholes model and Monte Carlo simulations. Further, we made use of four variance reduction methods to reduce the computational errors of Monte Carlo simulations. The accuracy of Monte Carlo simulations increases as the number of simulations increases but this comes at the cost of computational efficiency. The price calculated using BS model and the Monte Carlo simulations converges as the number of simulations increases but the change in error is not much significant. The variance reduction methods such as Antithetic variates and Control variates reduces error with very few number of simulations compared to simple Monte Carlo simulation. The Conditional Monte Carlo and Importance Sampling variance reduction methods reduces error whenever the difference between barrier level and the initial underlying stock price is higher. In other scenarios, they perform worse than the simple Monte Carlo simulation.

Overall, Monte Carlo simulation combined with variance reduction methods such as Antithetic variates and Control variates should be preferred over other methods. This paper focused only on the up-and-out call option and assumed all other barrier options work similarly. The further work will focus on applying the Antithetic variates and Control variates method to other barrier options as well. Also, the impact of barrier level on the Conditional Monte Carlo and Importance Sampling methods is yet to studied.

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